Ample Divisors, Automorphic Forms and Shafarevich's Conjecture

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Abstract. In this article we give a general approach to the following analogue of Shafarevich's conjecture for some polarized algebraic varieties; suppose that we fix a type of an algebraic variety and look at families of such type of varieties over a fixed Riemann surface with fixed points over which we have singular varieties, then one can ask if the set of such families, up to isomorphism, is finite.

In this paper we give a general approach to such types of problems. The main observation is the following; suppose that the moduli space of a fixed type of algebraic polarized variety exists and suppose that in some projective smooth compactification of the coarse moduli the discriminant divisor supports an ample one, then it is not difficult to see that this fact implies the analogue of Shafarevich's conjecture.

In this article we apply this method to certain polarized algebraic K3 surfaces and also to Enriques surfaces.

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1. Introduction

1.1. General Historical Review of the Problem. In his article published in the Proceedings of the International Congress of Mathematicians, Stockholm meeting held in 1962, Shafarevich wrote:

"One of the main theorems on algebraic numbers connected with the concept of discriminant is Hermit's theorem, which states that the number of extensions k'/k of a given degree and given discriminant is finite. This theorem may be formulated as follows: the number of extensions k'/k of a given degree whose critical prime divisors belong to a given finite set S is finite."

Inspired by this result of Hermit, Shafarevich conjectured: *"There exists only a finite number of fields of algebraic functions K/k of a given genus $g \geq 1$, the critical prime divisors of which belong to a given finite set S."* See page 290 of [40].

In unpublished work, Shafarevich proved his conjecture in the setting of hyperelliptic curves. On page 755 of [40], in his remarks on his papers, he wrote: *"Here two statements made in lecture are mixed into one: formulation of a result and conjecture. The result was restricted to the case of a hyperelliptic curves while conjecture concerned general curves...This conjecture became much more attractive after A. N. Parshin proved that implies Mordell conjecture...In 1983 it was proved by G. Faltings (Invent. Math.* ***73,*** 439-466(1983))*, with a proof of the Mordell conjecture as a consequence."*

In this paper we developed a general approach to higher dimensional analogues of Shafarevich's type conjectures over function field and apply this method to families of algebraic polarized K3 surfaces of special degrees or to families of polarized Enriques surfaces over an algebraic curve.

The formulation of the problem is the following. Let C be a fixed curve of genus g and let E be a set of different points on C. Let C be a fixed, non-singular algebraic curve, and let E be a fixed effective divisor on C such that all the points in E have multiplicity 1. Define Sh(C,E,Z) to be the set of all isomorphism classes of projective algebraic varieties $\mathcal{Z} \rightarrow C$ with a fibre of "type' Z such that the singular fibres are over the set E. The general Shafarevich type problem is: **"***For which "type" of varieties Z and data (C,E) is such that* Sh (C,E,Z) *is finite* ?

Previous work on Shafarevich-type problems include the following results. In the case when Z is a curve of genus $g > 1$ and E is empty, Parshin proved that Sh (C,E,Z) is finite, and Arakelov proved finiteness in the case E is not empty. See [2] and [35]. Faltings constructed examples showing that Sh (C,E,Z) is infinite for abelian varieties of dimension ≥ 8. See [14]. Saito an Zucker extended the construction of Faltings to the setting when Z is an algebraic polarized K3 surface. They were able to classify all cases when the set Sh (C,E,Z) is infinite. They are not considering polarized families. See [41]. Faltings proved the Shafarevich's conjecture over the number fields and thus he proved Mordell conjecture. Yves Andre proved the analogue of Shafarevich's conjectures over the number fields for K3 surfaces. See [1]. Using techniques from harmonic maps Jost and Yau analyzed Sh (C,E,Z) for a large class of varieties. See [23]. Ch. Peters studied finiteness theorems by considering variations of Hodge structures and utilizing differential geometric aspects of the period map and associated metrics on the period domain. See [36].

Our approach to prove finiteness of Sh (C,E,Z) is to analyze the discriminant locus $\mathcal{D}$ in some compactification of the coarse moduli space, i.e. those points which correspond to singular varieties in the Baily Borel compactification of the coarse moduli space of pseudo polarized algebraic K3 surfaces or Enriques surfaces. The main observation is that if $\mathcal{D}$ supports an ample divisor in some compactification, then Sh (C,E,Z) is finite.

The second author was informed that A. Parshin and E. Bedulev have proved finiteness for family of algebraic surfaces over a fixed algebraic curve assuming that all the fibres are non-singular.

Recently Migliorni, Kovács and Zhang that any family of minimal algebraic surfaces of general type over a curve of genus g and m singular points such that 2g-2+m$\leq$ 0 is isotrivial. This result was recently reproved by E. Bedulev and E. Viehweg. See [25], [26], [27], [28], [31], [34], [48], [49] and [12].

1.2. Organization of the Article. In **Section 2**, we define the terms needed to properly formulate the problem, and we prove general rigidity result from which the corresponding finiteness will follow. The main observation is that if the discriminant locus $\mathcal{D}$ in some compactification of the course moduli space of certain type of polarized

algebraic variety $\mathcal{M}$ is an ample divisor, then the moduli space $\mathfrak{M}(C;p_1,..,p_k)$ of maps of a pair of a fixed algebraic curve with fixed points $p_1,..,p_k$ on it $(C;p_{1,..,}p_k)$ to $\mathcal{M}$ such that the points $p_1,..,p_k$ are mapped to $\mathcal{D}$, then $\mathfrak{M}(C;p_1,..,p_k)$ is a discrete set. It is a well known fact that in order to prove finiteness of $\mathfrak{M}(C;p_1,..,p_k)$, one needs to prove that the volume of the image of C in $\mathcal{M}$ is bounded with respect to some metric on $\mathcal{M}$.

In **Section 3** we discuss various background material in the study of K3 surfaces following [3] and [19].

In **Section 4** we discuss various background material in the study Enriques surfaces following [21] and [22].

In **Section 5** we study the question when the discriminant locus in some compactification of the moduli space of polarized algebraic K3 surfaces is supporting an ample divisor. According to the Torelli Theorem and the epimorphism of the period map we know that the moduli space of algebraic polarized K3 surfaces is a locally symmetric space $\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}$, where $\mathfrak{H}_{2,19} = SO_0(2,19)/SO(2)\times SO(19)$ and $\Gamma_{K3,2d}$ is an arithmetic group acting on $\mathfrak{H}_{2,19}$. In [19] we proved that if Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}$ contains only one cusp of dimension 0, then there exists an automorphic form η_{2d} such that the support of the zero set of η_{2d} is exactly the closure $\overline{\mathcal{D}}$ of the discriminant locus $\mathcal{D}$.[1] Horikawa and the second author proved that for algebraic K3 surfaces with a polarization class e such that $\langle e,e\rangle = 2$ the Baily-Borel compactification $\overline{\Gamma_K 3, 2d\backslash\mathfrak{H}_2, 19}$ contains only one cusp of dimension 0. See [24]. Borcherds constructed in [8] a holomorphic automorphic form in case of degree two polarization based on idea of the second author.

In **Section 6** by using Gauss-Bonnet Theorem derive the following two Theorems:

THEOREM. Let C be an algebraic curve of genus g and $\pi : Y \to C$ be a three dimensional projective non-singular variety such that for every $t\in C\backslash E$, $\pi^{-1}(t) = X_t$ is a non-singular K3 surface and for each $t \in E$, $\pi^{-1}(t) = X_t$ is a singular surface. Suppose that on Y we have a polarization class H such that $H\,|_{X_t} = e$, <e,e>=2d and the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0. Let m_∞ be the number of points on C for which the local monodromy operator is of infinite order. Then the number of singular fibres of π is less or equal to $2g-2+m_\infty$.

THEOREM. Let C be an algebraic curve of genus g and $\pi : Y \to C$ be a three dimensional projective non-singular variety such that for every $t\in C\backslash E$, $\pi^{-1}(t) = X_t$ is a non-singular Enriques surface and for each $t \in E$, $\pi^{-1}(t) = X_t$ is a singular surface. Let m_∞ be the number of points on C for which the local monodromy operator is of infinite order. Then the number of singular fibres of π is less or equal to $2g-2+m_\infty$.

In **Section 7** by using the results of **Section 2** and the Theorem just formulated above we prove the finiteness of Sh (C,E,{K3,2}). By using the results of Saito and Zucker we prove the analogue of Shafarevich's conjecture for polarizations e such that $\langle e,e\rangle = 2d$ and the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0.

In **Section 8** we prove that Sh (C,E,Z) is finite when Z is an S-K3 surface, which means that for each $t\in C\backslash E$, $\pi^{-1}(t) = X_t$ is a K3 surface whose Picard Group$\subseteq S$, where S is a special primitive sublattice in $H^2(X,\mathbb{Z})$ of signature (1,k).

In **Section 9** we prove that Sh (C,E,Z) is finite when Z is an Enriques surface.

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[1] **1.** In [19] we refer to a Theorem of F. Scattone which is not correct. **2.** It was Nikulin who pointed out a mistake in an earlier version of [19].

*the invitation to work at MSRI. We want to thank S.-T. Yau for drawing our attention to [23]. Both authors acknowledge conversations with R. Borcherds, L. Katzarkov and T. Pantev on mathematical topics related to this paper. See [8]. We want to thank L. Katzarkov for his help with **Section 2**. The second author wants to thanks R. Borcherds for his valuable comments about an earlier version of this paper.*

2. Basic Rigidity Result

2.1. Introduction. In this section we shall prove a general Theorem about rigidity from which we shall derive finiteness results. The results in this section are established in very general context.

2.2. Basic Definitions and Facts about Moduli of Maps. Let S and X be two projective varieties. Let $f : S \to X$ be a morphism between them. Let $\Gamma_f \subset S \times X$ be the graph of the map $f : S \to X$. According to the results of Grothendieck the Hilbert scheme of $\Gamma_f \subset S \times X$ will be a projective scheme. See [39].

Definition 2. *We will denote the Hilbert scheme of $\Gamma_f \subset S \times X$ by $\mathfrak{M}_f(S, X)$ and called the moduli space of the map f.*

Grothendieck proved that $\mathfrak{M}_f(S, X)$ is a projective scheme. See [39].

Definition 3. a. *Let f:S→X be a morphism of projective varieties with dim S≤dim X and suppose that the morphism f:S→f(S) is finite. We say that f admits a non trivial one parameter deformation if there is a non-singular projective curve T and a family of algebraic maps F:T×S→X such that for some t_0 ∈T we have F_{t_0} =f and the morphism F:T×S→F((T×S) is finite too.*

b. *We will say that the deformation of f is trivial if F_t=f for all t∈*T.

c. *Two families of maps $F_1 : T\times S_1 \to X$ and $F_2 : T\times S_2 \to X$ are said to be isomorphic if there is a common finite cover S of S_1 and S_2 such that the lifts of F_1 to $T\times S$ and F_2 to $T\times S$ are isomorphic, meaning there exists a biholomorphic map $id\times g$ from $T\times S$ to itself such that F_1=$F_2 \circ (id \times g)$.*

Remark 4. *From now on we will consider only one parameter deformations of maps.*

2.3. The Rigidity Theorem.

Lemma 5. *Let f:S→X be a morphism between projective varieties S and X such that f:S→f(S) is a finite morphism, and let F be a deformation of f. Let D be an ample Cartier divisor on X, and assume that the image of S is not contained in D. Suppose that $F^*(D)$=$D_S\times T$ where D_S is a Cartier divisor on S, then F is the trivial deformation.*

PROOF: Choose sufficiently large n such that nD is a very ample divisor on X. Since Definition 3 implies that the map F is a finite map we can conclude that $\mathrm{F}^*(\mathrm{nD})=\mathrm{nF}^*(\mathrm{D})$ will be a very ample divisor on S×T. From the condition that $\mathrm{F}^*(\mathrm{D})=\mathrm{D}_S\times\mathrm{T}$ we deduce that the line bundle $\mathcal{O}_{\mathrm{S}\times\mathrm{T}}$ $(\mathrm{F}^*(\mathrm{nD}))$ is isomorphic to $\mathcal{O}_{\mathrm{T}} \otimes \mathcal{O}_{\mathrm{S}}(n\mathrm{D}_S)$. So we have

$$H^0(\mathrm{S}\times\mathrm{T},\mathcal{O}_{\mathrm{S}\times\mathrm{T}}\ (\mathrm{F}^*(\mathrm{nD})))=H^0(\mathrm{S}\times\mathrm{T},\mathcal{O}_{\mathrm{T}} \otimes \mathcal{O}_{\mathrm{S}}(n\mathrm{D}_S)) = H^0(\mathrm{T},\mathcal{O}_{\mathrm{T}}) \otimes H^0(\mathrm{S},\mathcal{O}_{\mathrm{S}}(n\mathrm{D}_S)).$$

From here we obtain that any section $\sigma \in H^0(\mathrm{S}\times\mathrm{T},\mathcal{O}_{\mathrm{S}\times\mathrm{T}}\ (\mathrm{F}^*(\mathrm{nD})))=H^0(\mathrm{T},\mathcal{O}_{\mathrm{T}})\otimes H^0(\mathrm{S},\mathcal{O}_{\mathrm{S}}(n\mathrm{D}_S))$ can be written in the form $\sigma=\sum_i c_i\otimes\pi_2^*(\sigma_i)$ where c_i are constants and σ_i are sections of $\mathcal{O}_{\mathrm{S}}(\mathrm{D}_S)$ on S. So we can conclude that the projective morphism defined by the choice of the basis in $H^0(\mathrm{S}\times\mathrm{T},\mathcal{O}_{\mathrm{S}\times\mathrm{T}}\ (\mathrm{F}^*(\mathrm{nD})))$ of the very ample line bundle $\mathcal{O}_{\mathrm{S}\times\mathrm{T}}$ $(\mathrm{F}^*(\mathrm{nD}))$ will map the subvariety s$\times T$ to a point. This contradicts the definition of the very ample line bundle, which implies that any basis in $H^0(\mathrm{S}\times\mathrm{T},\mathcal{O}_{\mathrm{S}\times\mathrm{T}}\ (\mathrm{F}^*(\mathrm{nD})))$ will define an embedding of S×T into some projective variety. ■.

Theorem 6. *Let S and X be projective varieties, and let D be an ample Cartier divisor on X. Let $\mathcal{F}$ be the set of all finite morphisms f:S→X such that* **a.** *f(S) is not contained in D,* **b.** *the map f:S→f(S) is finite* **c.** *$f^{-1}(D)=D_S$, where D_S is a fixed divisor on S for all f∈ $\mathcal{F}$.* **d.** *Suppose that vol(f(S))<c, where the volume of f(S) is with respect to the restriction of Fubini Study metric on f(S) obtained from the embedding of X in some projective space given by the very ample divisor n_0D, then $\mathcal{F}$ is a finite set.*

PROOF: Let us choose n sufficiently parge positive integer such that nD is a very ample divisor on X. By assumption, the divisor $\mathcal{D}=n\mathrm{f}^*(\mathrm{D})\times\mathrm{X}+\mathrm{S}\times\mathrm{nD}$ is an ample divisor on S×X since f is a finite. Let us denote by $|\mathcal{D}|$ the linear system defined by the ample divisor $\mathcal{D}$. Let us fix a finite map f:S→X such that f fulfils conditions **a, b** and **c.**

Definition 7. *Once we fix the map f as above we define the set $\mathcal{M}_{f,D}$ as the subset of deformation space $\mathfrak{M}_f(S,X)$ of the fixed map f as defined in* ***Definition 2****, which fulfils conditions* **a, b** *and* **c** *of Theorem 6.*

Proposition 8. *$\mathcal{M}_{f,D}$ is a finite set.*

PROOF: Since f is a finite map it is easy to see that the linear system $|\mathcal{D}|$ defines a finite projective map $\phi_{|\mathcal{D}|}$:S×X→ $\mathbb{P}^N$ once we choose a basis in $H^0(\mathrm{S}\times\mathrm{X},\mathcal{O}_{\mathrm{S}\times\mathrm{X}}(\mathcal{D}))$. Let $\Gamma_{\mathrm{f}}\subset$S×X be the graph of the map f:S→X and let $\mathfrak{M}_f(S,X)$ be the projective variety defined in Definition 2. From the results in [39] we deduce that the condition that f(S) is not contained in the fixed Cartier divisor D define a Zariski open set in $\mathfrak{M}_f(S,X)$ which we will denote by $\mathfrak{M}_{f,D}$. On the other hand the condition that f∈ $\mathfrak{M}_{f,D}$ and f*(D) is a fixed divisor D_S in S is a closed condition, i.e. the set $\mathcal{M}_{f,D}:=\{\mathrm{f}\in\mathcal{M}_D|\mathrm{f(D)}\subsetneq\mathrm{D}$ and $\mathrm{f}^*(\mathrm{D})$ is a fixed divisor D_S in S$\}$ is a closed subscheme in $\mathfrak{M}_{f,D}$. Lemma 5 implies that the set $\mathcal{M}_{f,D}$ is zero dimensional. Since $\mathcal{M}_{f,D}$ is a quasi-projective, zero dimensional scheme, then as a set $\mathcal{M}_{f,D}$ is a finite set. Proposition 8 is proved. ■.

It remains to show that the set $\mathcal{Z}_D$ is a finite set where $\mathcal{Z}_D$ is the union of all $\mathcal{M}_{f,D}$, over all maps f which fulfil the conditions **a, b, c,** and **d** .

Proposition 9. *$\mathcal{Z}_D$ is a finite set.*

PROOF: Define a height on $\mathcal{Z}$ in the following manner. Let $\phi_{|n_0D|}:X\subset\mathbb{P}^m$ be the embedding given by the linear system $|n_0D|$ of the very ample divisor n_0D on X. Let $\omega_X:=\phi^*_{|n_0D|}(\omega_{\mathbb{P}^m})$, where $\omega_{\mathbb{P}^m}$ be the Fubini-Study form. Each point of $\mathcal{Z}$ is represented by a map $f:S\to X$ which satisfies conditions **a, b,** and **c** of Theorem 6**.** Define the height function h on $\mathcal{Z}$ as follows:

$$h(f)=\int_{f(S)}\wedge^k(\omega_{\mathbb{P}^m})=vol(f(S))=(n_0)^k<D,..,D>\Big|_{f(S)}$$

where k is the dimension of S. Since f is a finite morphism, then f(S) will have also dimension k. Since h is defined as an intersection number, h is integer valued, hence h is a locally constant function, i.e. constant on each connected component of $\mathcal{Z}$. Condition **d** implies that h is a bounded function.

We will prove now that $\mathcal{Z}_D$ is a compact set. Since $\mathcal{Z}_D$ is a discrete set, therefore $\mathcal{Z}_D$ will be finite. We need to prove that from any sequence $\{f_n\}$ in $\mathcal{Z}$ there is a subsequence which converges weakly to an algebraic map, i.e. the corresponding subsequence of images of S converges to an algebraic subvariety of X. Bishop's theorem implies that. See page 292 of [7] or page 321 of [18]. Proposition 9 is proved. ■.

From Proposition 9 Theorem 6 follows directly. ■.

Remark 10. *Our method of proof of Shafarevich-type problems for varieties over function fields utilizes the results of this section in the following manner. In order to prove finiteness of Sh(C,E,Z), we let X be the course moduli space of varieties of type Z. Let $\overline{X}$ be a compactification of X such that D=$\overline{X}\setminus X$ is a divisor of normal crossings. It is then necessary to show that D supports an ample divisor on $\overline{X}$. We then study maps of C into $\overline{X}$ with the requirement that the subset of C which intersects D is exactly E. Theorem 6 yields the desired finiteness result.*

3. Basic Properties of K3

We will review some basic properties of algebraic K3 surfaces. For a more general and complete discussion, the reader is referred to [3] and [10].

3.1. Definition of a K3 Surface. A K3 surface is a compact, complex two dimensional manifold with the following properties: **i.** There exists a non-zero holomorphic two form ω on X. **ii.** $H^1(\mathrm{X},\mathcal{O}_{\mathrm{X}}) = 0$.

Remark 11. *For the purposes of this article, we will assume that all surfaces are projective varieties.*

From the defining properties, one can prove that the canonical bundle on X is trivial. In [3] and [10], the following topological properties are proved. The surface X is simply connected, and the homology group $H_2(X,\mathbb{Z})$ is a torsion free abelian group of rank 22. The intersection form $< \, , \, >$ on $H_2(X,\mathbb{Z})$ has the properties: **i.** $<$u,u$>$= 0 mod(2);**ii.** $\det(\langle e_i, e_j\rangle) = -1$ and **iii.** the symmetric form $< \, , \, >$ has a signature $(3,19)$.

Theorem **5** on page 54 of [38] implies that as an Euclidean lattice $H_2(X,\mathbb{Z})$ is isomorphic to the K3 lattice Λ_{K3}, where

$$H_2(X,\mathbb{Z}) \backsimeq \Lambda_{K3} = \mathbf{H}^3 \oplus (-E_8)^2$$

with

$$\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

being the hyperbolic lattice. Let $\alpha = \{\alpha_i\}$ be a basis of $H_2(X,\mathbb{Z})$ with intersection matrix Λ_{K3}. The pair (X,α) is called a marked K3 surface. Let $e \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ be the class of a hyperplane section, i.e. an ample divisor. The triple (X,α,e) is called a marked, polarized K3 surface. The degree of the polarization is an integer $2d$ such that $< e, e >= 2d$.

3.2. Moduli of Marked, Algebraic and Polarized K3 surfaces. From [37] and [29] we have that the moduli space of isomorphism classes of marked, polarized, algebraic K3 surfaces of a fixed degree $2d$, which we denote by $\mathcal{M}^{2d}_{K3,mpa}$, is equal to an open set in the symmetric space $\mathfrak{h}_{2,19} := SO_0(2,19)/[SO(2)\times SO(19)]$. Let

$$\Gamma_{K3,2d} = \{\phi \in Aut(\Lambda_{K3})| < \phi(u), \phi(u) >=< u, u > \text{ and } \phi(u) = u\}.$$

The moduli space of isomorphisms classes of polarized, algebraic K3 surfaces of a fixed degree 2d, which we denote by $\mathcal{M}^{2d}_{K3,pa}$ is isomorphic to a Zariski open set in the quasi-projective variety $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$.

If we allow our surface to have singularities which are at most double rational points, then the corresponding moduli space of polarized, algebraic surfaces is equal to the entire locally symmetric space $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$. In other words, marked and pseudo-polarized surfaces corresponding to points in the complement of $\mathcal{M}^{2d}_{K3,mpa}$ in $\mathfrak{h}_{2,19}$, are those surfaces for which the projective image corresponding to any power of the polarization is singular with singularities which are double rational points. Specifically, the relation between $\mathcal{M}^{2d}_{K3,mpa}$and $\mathfrak{h}_{2,19}$ is through the period map, which we now will describe. The period map π for marked K3 surfaces (X,α) is defined by integrating the holomorphic two form ω along the basis α of $H_2(X,\mathbb{Z})$, meaning

$$\pi(X,\alpha) := (..., \int_{\alpha_i}\omega, ...) \in \mathbb{P}^{21}.$$

The Riemann bilinear relations hold for $\pi(X,\alpha)$, meaning

$$\langle \pi(X,\alpha), \pi(X,\alpha)\rangle = 0 \text{ and } \left\langle \pi(X,\alpha), \overline{\pi(X,\alpha)}\right\rangle > 0.$$

Choose a primitive vector $e \in H^2(X,\mathbb{Z})$ such that $\langle e, e\rangle = 2d > 0$. As in [37], one has the description of $\mathfrak{h}_{2,19}$ as one of the open sets of the quadric Q defined in $\mathbb{P}(\Lambda_{K3}\otimes\mathbb{C})$ by the equations $\langle u, u\rangle = 0$ and $\langle u, e\rangle = 0$ and the inequality $\langle u, \overline{u}\rangle > 0$. Results from [29] and [37] combine to prove that the period map π is surjection, i.e. each point of the period domain $\mathfrak{h}_{2,19}$ corresponds to a marked pseudo-polarized algebraic K3 surface. By pseudo-polarized algebraic K3 surface we understand a pair (X,e) where e corresponds to either ample divisor or pseudo ample divisor, which means that for any effective divisor D in X, we have $\langle D, e\rangle \geq 0$. Mayer proved the linear system $|3e|$ defines a map:

$$\phi_{|3e|} : X \to X_1 \subset \mathbb{P}^m$$

such that: **i.** X_1 has singularities only double rational points. **ii.** $\phi_{|3e|}$ is a holomorphic birational map. From the result of Donaldson and the surjectivity of the period map, it follows that the moduli space of pseudo-polarized algebraic K3 surfaces of degree 2d $\mathcal{M}^{2d}_{K3,ppa}$ is isomorphic to the locally symmetric space $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$. See [13]. For the discussion of the global Torelli Theorem for polarized, algebraic K3 surfaces see [37], [3] and [10].

3.3. Description of the Discriminant Locus in the Moduli Space of Pseudo-Polarized Algebraic K3 Surfaces.

Notation 12. *From now on if a set Y is contained in X, then the complement of Y in X will be denoted by $X \ominus Y$.*

The complement of $\mathcal{M}^{2d}_{K3,mpa}$ in $\mathfrak{h}_{2,19}$ can be described as follow. Given a polarization class $e \in \Lambda_{K3}$, set T_e to be the orthogonal complement to e in Λ_{K3}, i.e. T_e is the transcendental lattice. Then we have the realization of $\mathfrak{h}_{2,19}$ as one of the components of

$$\{u \in \mathbb{P}(T_e \otimes \mathbb{C}) |\, \langle u, u\rangle = 0 \text{ and } \langle u, \overline{u}\rangle > 0\}.$$

Define the set $\Delta(e) := \{\delta \in \Lambda_{K3} |\, \langle e, \delta\rangle = 0 \text{ and } \langle \delta, \delta\rangle = -2\}$. For each $\delta \in \Delta(e)$, define the hyperplane

$$H(\delta) = \{u \in \mathbb{P}(T_e \otimes \mathbb{C}) |\, \langle u, \delta\rangle = 0\}.$$

Let

$$\mathcal{H}_{K3,2d} = \underset{\delta\in\Delta(e)}{\cup}(H(\delta) \cap\, \mathfrak{h}_{2,19}).$$

Set $\mathcal{D}^{2d}_{K3} := \Gamma_{K3,2d}\backslash\mathcal{H}_{K3,2d}$. Results from [30], [37], [43] and [29] imply that $\mathcal{D}^{2d}_{K3}$ is the complement of the moduli space of algebraic polarized K3 surfaces $\mathcal{M}^{2d}_{K3,pa}$ in the locally symmetric space $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$, i.e. $\mathcal{D}^{2d}_{K3} = (\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}) \ominus \mathcal{M}^{2d}_{K3,pa}$.

4. Enriques Surfaces and Their Moduli

We will define an Enriques surface Y to be X/ρ, where X is a K3 surface and ρ is an involution acting on X without fixed points. On $\Lambda_{K3} \approxeq \mathbf{H}^3 \oplus (-E_8)^2$ we will define the Enriques involution $\rho(z_1 \oplus z_2 \oplus z_3 \oplus x \oplus y) = (-z_1 \oplus z_3 \oplus z_2 \oplus y \oplus x)$. Let Λ^+_{K3} and Λ^-_{K3} be the $\rho-$invariant and $\rho-$anti-invariant sublattices. The unimodular lattice $\frac{1}{2}\Lambda^+_{K3}$ is isometric to the Enriques lattice Λ_{Enr}. We define the space $\Omega_{Enr} = \mathbf{P}(\Lambda^-_{K3} \otimes \mathbf{C}) \cap \Omega_{K3}$, where Ω_{K3} is the period domain for marked K3 surfaces. It is easy to see that

$$\Omega_{Enr} := SO_0(2, 10)/SO(2) \times SO(10) = \mathfrak{h}_{2,10}.$$

Definition 13. *We define* $\Gamma_{Enr} = rest_{\Lambda^-_{K3}}\{g \in Aut(\Lambda_{K3}) | g \circ \rho = \rho \circ g\}$.

Definition 14. *We defined* $\widetilde{\Gamma}_{Enr}$ *as a subgroup of finite index in* Γ_{Enr} *which preserves the so called* $\mathbf{H}-$*marking of the Enriques surfaces, which means a pair (Y,j) and*

$$\text{j:}\mathbf{H} \rightarrow H^2(Y, \mathbb{Z})_f.$$

Remark 15. *It was proved in [10] that* $\widetilde{\Gamma}_{Enr}$ *as a subgroup of finite index in* Γ_{Enr}.

Let $\Delta_+ := \{\delta \in \Lambda^-_{K3} | < \delta, \delta >= -2\}$ and $\Delta_- := \{\delta \in \Lambda_{K3} | < \delta, \delta >= -2 \text{ and } \delta^\rho \neq \delta\}$. It is shown on p. 283 of [10] that no point of the hyperplane $H_l = \{p \in \Omega_{Enr} | < p, \delta >= 0$ and $\delta \in \Delta_+\}$ can be the period of a marked Enriques surfaces. Namikawa showed in [32] the following result. Let $\delta \in \Delta_-$, then the points of $H_\delta = \{p \in \Omega_{Enr} | < p, \delta - \delta^\rho >= 0\}$ corresponds to Enriques surfaces with double rational points.

Definition 16. *We will define two divisors* $\mathcal{D}_+ := \widetilde{\Gamma}_{Enr} \backslash \cup_l H_l$ *(for all* $l \in \Delta_-$*) and* $\mathcal{D}_- := \widetilde{\Gamma}_{Enr} \backslash \cup_\delta H_\delta$ *in* $\mathcal{M}^2_{Enr,\mathbf{H}}$*, (for all* $\delta \in \Delta_-$*).*

The following Theorems are due to Horikawa. See [10].

Theorem 17. *The coarse moduli space of* $\mathbf{H}-$*marked Enriques surface* $\mathcal{M}^2_{Enr,\mathbf{H}}$ *is isomorphic to* $\left(\widetilde{\Gamma}_{Enr}\backslash\Omega_{Enr}\right)\setminus(\mathcal{D}_+\cup\mathcal{D}_-)$.

Theorem 18. *The isomorphism class of Enriques surface is uniquely determined by its period in* $\Gamma_{Enr}\backslash\Omega_{Enr}$.

Remark 19. *The results of Borcherds in [9], imply that both divisors* $\mathcal{D}_+$ *and* $\mathcal{D}_-$ *are irreducible in* $\mathcal{M}^2_{Enr,\mathbf{H}}$ *and so their closures* $\overline{\mathcal{D}_+}$ *and* $\overline{\mathcal{D}_-}$ *in the Baily-Borel compactification* $\overline{\widetilde{\Gamma}_{Enr}\backslash\Omega_{Enr}}=\overline{\widetilde{\Gamma}_{Enr}\backslash\mathfrak{h}_{2,10}}$ *of* $\widetilde{\Gamma}_{Enr}\backslash\mathfrak{h}_{2,10}$.

5. Discriminant as an Ample Divisor in the Moduli of Certain Pseudo-Polarized Algebraic K3 Surfaces.

We will assume from now on that the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$ contains only one dimensional cusp. We know that this is the case when d=1. See [24]. Our proof that in this case the closure $\overline{\mathcal{D}^{2d}_{K3}}$ of $\mathcal{D}^{2d}_{K3}$ in the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$ contains the support of an ample divisor in $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ will proceed in two steps. **Step 1.** We recall the construction of an ample line bundle $\mathcal{L}^{-1}_{K3,2d}$ on $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$ by means of the factor of automorphy given by the functional determinant of the group action of $\Gamma_{K3,2d}$ on $\mathfrak{h}_{K3,2d}$. Holomorphic sections of $\left(\mathcal{L}^{-1}_{K3,2d}\right)^{\otimes n}$ are automorphic forms on $\mathfrak{h}_{K3,2d}$ with respect to the group $\Gamma_{K3,2d}$. **Step 2.** We recall work from [19] which constructs an automorphic form on $\mathfrak{h}_{K3,2d}$ with respect to the group action $\Gamma_{K3,2d}$, and which vanishes on a divisor $\mathcal{E}^{2d}_{K3}$ whose support is contained in $\mathcal{D}^{2d}_{K3}$.

5.1. Construction of a Line Bundle on $\mathfrak{h}_{K3,2d}$.

Let us consider the principle bundle SO(2) bundle SO(2)→SO(2,19)/SO(19)→ $\mathfrak{h}_{K3,2d}$. Since SO(2)= $U(1)$ we can associate a complex line bundle $\mathcal{L}_{\Gamma_{K3,2d}}\to\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$. More explicitly, let $\gamma\in\Gamma_{K3,2d}$ and set

$$\mathrm{j}(\gamma,\tau):=\det\left(\frac{\partial\gamma(\tau)}{\partial\tau}\right)^{-19}.$$

Then $\mathcal{L}_{\Gamma_{K3,2d}}$ is constructed via the factor of automorphy $\mathrm{j}(\gamma,\tau)$, i.e. $\mathcal{L}_{\Gamma_{K3,2d}}$ is obtained as the quotient of $\mathfrak{h}_{K3,2d}\times\mathbb{C}$ through the identification $(\tau,w)\sim(\gamma\tau,j(\gamma,\tau)w)$. See [46]. From [5] and [6], we have that $\mathcal{L}^{-1}_{\Gamma_{K3,2d}}$ extends to an ample bundle on the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$, and automorphic forms on $\mathfrak{h}_{K3,2d}$ with respect to $\Gamma_{K3,2d}$ corresponds to sections of $\left(\overline{\mathcal{L}^*_{\Gamma_{K3,2d}}}\right)^{\otimes n}$, where $\overline{\mathcal{L}^*_{\Gamma_{K3,2d}}}$ is the extension of the dual of the line bundle $\mathcal{L}_{\Gamma_{K3,2d}}$ to the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ for some positive integer n. We have proved in [19] the following Lemma:

Lemma 20. *Let* $\pi:\mathcal{X}\to\mathcal{M}^{2d}_{K3,pa}$ *be the versal family of polarized algebraic K3 surfaces of degree 2d, then* $\pi_*(\mathcal{K}_{\mathcal{X}/\mathcal{M}^{2d}_{K3,pa}})\approxeq\mathcal{L}_{\Gamma_{K3,2d}}$.

The following Theorem from [19] is the main result of this section

Theorem 21. *Let $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ be the Baily-Borel compactification of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}$ such that it contains only one zero dimensional cusp, then there is an ample divisor $\overline{\mathcal{E}^{2d}_{K3}}$ on $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ with support contained in the closure of $\mathcal{D}^{2d}_{K3}$ in $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}$ which we will denote by $\overline{\mathcal{D}^{2d}_{K3}}$ such that $\mathcal{L}^{-1}_{\Gamma_{K3,2d}} \approxeq \mathcal{O}_{\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}}(\mathcal{E}^{2d}_{K3})$.*

PROOF: Our proof of Theorem 20 involves an explicit constriction of a holomorphic function on $\mathfrak{h}_{2,19}$ which is modular with respect to $\Gamma_{K3,2d}$ and which is non-vanishing on $\mathcal{M}^{2d}_{K3,pa}$ and we know that $\mathcal{M}^{2d}_{K3,pa} = (\Gamma_{K3,2d}\backslash\mathfrak{h}_{K3,2d}) \ominus \mathcal{D}^{2d}_{K3}$.

Since Baily-Borel bundle $\mathcal{L}^{-1}_{\Gamma_{K3,2d}}$ can be prolonged to an ample bundle $\overline{\mathcal{L}^{-1}_{\Gamma_{K3,2d}}}$ over the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$, by the results in [5] and [6] and since our form is a section of a power of $\mathcal{L}^{-1}_{\Gamma_{K3,2d}}$, by Lemma 20 hence Theorem 21 follows.$\blacksquare$

Let (X,e) be a polarized K3 surface of degree 2d, and let $\mathcal{T}_{(X,e)}$ be the sheaf of holomorphic vector fields on (X,e). From Kodaira-Spencer deformation theory, we can identify the tangent space $T_{\mathcal{M}^{2d}_{K3,mp}}$ at the point (X,e) with $H^1(X,\mathcal{T}_{(X,e)})$. The existence of the holomorphic two form ω on X implies that we can identify $H^1(X,\mathcal{T}_{(X,e)})$ with $H^1(X,\Omega^1)$, where Ω^1 is the sheaf of holomorphic one forms on X. One can deduce that the tangent space $T_{\mathcal{M}^{2d}_{K3,mp}}$ to the moduli space $\mathcal{M}^{2d}_{K3,mp}$ at the point (X,α, e) can be identified with the space

$$H^1(X,\Omega^1)_0 = \{u \in H^1(X,\Omega^1)| < u, e >= 0\}.$$

We view any $\phi \in H^1(X,\mathcal{T}_{(X,e)})$ as a linear map from $\Omega^{1,0}$ to $\Omega^{0,1}$ pointwise on X. Given ϕ_1 and ϕ_2 in $H^1(X,\mathcal{T}_{(X,e)})$, the trace of the map $\phi_1 \circ \overline{\phi_2} : \Omega^{0,1} \to \Omega^{0,1}$ at a point a point x∈X with respect to the unit volume CY metric g (meaning a Kähler-Einstein metric compatible with the given polarization class e) is simply

$$Tr(\phi_1 \circ \overline{\phi_2}) = \sum_{k,l,m,n} (\phi_1)^k_l\, \overline{(\phi_2)^m_n}\, g^{n\bar{l}} g_{k\overline{m}}.$$

The existence of a Calabi-Yau metric on X compatible with the polarization e (unique up to a scale) is guaranteed by Yau's Theorem [47]. We define Weil-Petersson metric on $\mathcal{M}^{2d}_{K3,mpa}$ via the inner product

$$\langle\phi_1,\phi_2\rangle := \int_X Tr(\phi_1 \circ \overline{\phi_2}) vol_g.$$

on the tangent space of $\mathcal{M}^{2d}_{K3,mpa}$ at (X,α, e). It is shown in [44] that the Weil-Petersson metric is equal to the Bergman metric on $\mathfrak{h}_{2,19}$. Therefore, the Weil-Petersson metric is a Kähler metric with a Kähler form μ_{WP}.

Since $\mathfrak{h}_{2,19}$ is simply connected and over the moduli space of marked polarized algebraic K3 surfaces $\mathcal{M}^{2d}_{K3,mp} \subset \mathfrak{h}_{K3,2d}$ we have a universal family of marked and polarized K3 surfaces $\mathcal{X}^{2d} \to \mathcal{M}^{2d}_{K3,mp}$, there exists a non-vanishing holomorphically varying family of holomorphic two forms over $\mathfrak{h}_{K3,2d}$. For any such family consider the function on $\mathfrak{h}_{K3,2d}$ defined by

$$\|\omega\|^2_{L^2} = \langle\omega,\omega\rangle = \int_X \omega \wedge \overline{\omega}.$$

In [44] and [42] it was proved that $\log\|\omega\|^2_{L^2}$ is a potential for the Weil-Petersson metric. The following result from [19] proves the existence of a second potential for the Weil-Petersson metric.

Theorem 22. *Let (X,e) be a polarized, algebraic K3 surface of degree 2d, and let μ denote the unit volume Calabi-Yau form on X which is compatible with the polarization class e. Let $\{\omega\}$ be a non-vanishing, holomorphically varying family of holomorphic two form on $\mathfrak{h}_{2,19}$.* **A.** *Let $\det\Delta_{(X,e)}$ denote the zeta regularized product of the non-zero eigenvalues of the Laplacian of the CY metric which acts on the space of smooth functions on X. Then*

$$dd^c\log\left(\frac{\det\Delta_{(X,e)}}{\|\omega\|^2_{L^2}}\right)=0,$$

or equivalently

$$-dd^c\log\left(\det\Delta_{(X,e)}\right)=-dd^c\log\left(\|\omega\|^2_{L^2}\right)=\mu_{WP}.$$

in other words $-\log\left(\det\Delta_{(X,e)}\right)$ is a potential for the Weil-Petersson metric on $\mathcal{M}^{2d}_{K3,mpa}$.

B. *There is a holomorphic function (possibly multi-valued) $f_{K3,\omega,2d}$* on $\mathfrak{h}_{2,19}\backslash\mathcal{M}^{2d}_{K3,mpa}$ *such that*

$$|\mathrm{f}_{K3,\omega,2d}|^2=\left(\frac{\det\Delta_{(X,e)}}{\|\omega\|^2_{L^2}}\right);$$

hence $\mathrm{f}_{K3,\omega,2d}$ *does not vanish on $\mathcal{M}^{2d}_{K3,mpa}$.*

The reader is referred to [19] for details of the proof of Theorem 22.

Important Note. The function $\mathrm{f}_{K3,\omega,2d}$ constructed in Theorem 22 part **B** is possibly multi-valued function with a divisor contained in the complement of $\mathcal{M}^{2d}_{K3,mpa}$ in $\mathfrak{h}_{2,19}$. At this point, we do not assert any behavior of $\mathrm{f}_{K3,\omega,2d}$ with respect to the discrete group $\Gamma_{K3,2d}$. Theorem 22 is valid for any degree 2d of the polarization class e.

If we want to conclude automorphic behavior of the function $\mathrm{f}_{K3,\omega,2d}$ we need to have that $\|\omega\|^2_{L^2}$ is a meromorphic or holomorphic automorphic form since we know that $\det\Delta_{(X,e)}$ is a function on $\mathcal{M}^{2d}_{K3,pa}=\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$.

It is easy to see that we can always construct a meromorphic section $\{\omega\}$ of the Baily-Borel line bundle $\mathcal{L}^{-1}_{K3,2d}$. Then $\mathrm{f}_{K3,\omega,2d}$ will have additional zeroes and poles coming from the poles and the zeroes of the meromorphic section $\{\omega\}$ of the Baily-Borel line bundle $\mathcal{L}^{-1}_{K3,2d}$.

In order to conclude that some power of $\mathrm{f}_{K3,\omega,2d}$ is an automorphic form on $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ we need to construct a holomorphic family of non vanishing holomorphic two forms ω over $\mathfrak{h}_{2,19}$ such that $\|\omega\|^2_{L^2}$ is automorphic form of weight -2. In order to construct such forms we will use the special ,polarization 2d for which the Baily Borel compactification of the moduli space of pseudo polarized K3 surface contain a unique cusp of dimension 0. The analogue of the non vanishing family of holomorphic forms in case of elliptic curves is the family of the so called normalized one forms ω on the upper half plane $\mathfrak{h}:=\{\tau\in\mathbb{C}|\operatorname{Im}\tau>0\}$ such that

$$\operatorname{Im}\tau=\|\omega\|^2_{L^2}=\frac{-\sqrt{-1}}{2}\int_E\omega\wedge\overline{\omega}.$$

Remark 23. *Theorem 22 is a generalization of a known result which exists in the setting of elliptic curves. A generalization of Theorem 22 in the setting of CY manifolds was established in [20] and recently in [45].*

5.2. Construction of a Special Family of Holomorphic Two Forms for Some Polarizations. Our construction of the holomorphic family of normalized holomorphic two forms in the case when the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0.

One can construct a normalized family of forms $\{\omega_{2d}\}$ as follows: **Step 1.** Let $\tau_\infty \in \overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ is the unique zero dimensional cusp. Since $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ is a projective variety, we can find a disk D$\subset$ $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ such that $\tau_\infty \in D$ and D$\backslash\tau_\infty \subset \Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$. By restricting the versal family $\pi_{2d} : \mathcal{X}^{2d} \to \mathcal{M}^{2d}_{K3,mp}$ to the punctured disk D*=D$\backslash\tau_\infty$ we obtain a family of algebraic K3 surfaces $\mathcal{X}_{D^*} \to D^*$. The monodromy operator T acting on $H^2(X_t,\mathbb{Z})$ is such that $(T^m - id)^3 = 0$ and $(T^m - id)^2 \neq 0$. By taking a finite covering of D^* we may assume that $(T - id)^3 = 0$ and $(T - id)^2 \neq 0$. **Step 2.** The family $\mathcal{X}_{D^*} \to D^*$ constructed in **Step 1** defines up to a sign a unique cycle γ up to the action of the automorphisms group of the primitive cycles such that $T\gamma = \gamma$ and there exists cycles μ and η such that $T\mu = \mu+\gamma$ and $T\eta = \mu+\eta+\gamma$. **Step 3.**Since $\mathfrak{h}_{2,19}$ is a contractible, there exists a globally defined, non-vanishing, holomorphically varying family of holomorphic two forms, i.e.

$$\omega_\tau \in H^0(\mathfrak{h}_{2,19}, \pi_*\mathcal{K}_{\mathcal{X}^{2d}_{K3}/\mathcal{M}^{2d}_{K3,mpp}})$$

where $\mathcal{K}_{\mathcal{X}^{2d}_{K3}/\mathcal{M}^{2d}_{K3,mpp}}$ is the relative canonical sheaf. **Step 4.** The following Lemma is true:

Lemma 24. *The function $\phi(\tau) := \int_\gamma \omega_\tau$ is non vanishing on $\mathfrak{h}_{2,19}$.*

PROOF: Suppose that at some point $\tau_0 \in \mathfrak{h}_{2,19}$ $\phi(\tau_0) = 0$. From the epimorphism of the period map proved in [43] it follows that τ_0 corresponds to the periods of some marked pseudo polarized K3 surface (X$_0, \alpha, e$). It is easy to see that $\gamma \in T_e$, i.e. $\langle\gamma, e\rangle = 0$. This implies that γ can be realized as an algebraic cycle in the K3 surface X$_0$. Indeed we can find a line bundle $\mathcal{L}$ on X$_0$ such that the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ will be γ. Then the poles and zeroes of any meromorphic section of $\mathcal{L}$ will give a realization of γ as an algebraic cycle. A Theorem proved in [37] states that after finite number of reflections generated by vectors $\delta \in \Delta(e) \cap H^{1,1}(X_0,\mathbb{R})$ we may assume that γ can be realized as an elliptic curve E embedded in X$_0$. On the other hand side the condition $\langle\gamma, e\rangle = 0$ implies that $\langle E, E\rangle = \langle E, e\rangle = 0$. So from here it follows that the Elliptic curve should be contracted by the map $\phi_{|3e|} : X_0 \to \mathbb{P}^m$ defined by the linear system $|3e|$. By a theorem of Grauert it is possible if and only if $\langle E, E\rangle < 0$. See [15]. So we obtain a contradiction. Lemma 24 is proved. ■.

Definition 25. *We define the normalized family of holomorphic two forms as* $\{\omega_{n,2d}\} := \left\{\frac{\omega_\tau}{\phi(\tau)}\right\}$.

Following the identical steps in the case of elliptic curves, we construct a family of forms ω such that

$$\operatorname{Im}\tau = \|\omega\|^2_{L^2} = \frac{-\sqrt{-1}}{2}\int_E \omega\wedge\overline{\omega}.$$

Definition 26. *For any g$\in$ $\Gamma_{K3,2d}$ and K3 surface X represented by the period point $\tau \in \mathfrak{h}_{2,19}$, we set*

$$\psi(g,\tau)=\int_\gamma g^*\omega_{n,2d}$$

where γ denotes the invariant vanishing cycle.

Remark 27. *$\psi(g,\tau)$ defines one cocycle with respect to the group $\Gamma_{K3,2d}$ with coefficients in the $\Gamma_{K3,2d}$ module of invertible analytic functions $\mathcal{O}^*_{\mathfrak{h}_{2,19}}$ on $\mathfrak{h}_{2,19}$, i.e. $\{\psi(g,\tau)\}\in H^1(\Gamma_{K3,2d},\mathcal{O}^*_{\mathfrak{h}_{2,19}})$.*

Lemma 28. *When the degree of the polarization is such that there is a single zero-dimensional cusp at the boundary of the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$, then the cocycle $\psi(g,\tau)$ defines a line bundle on $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ isomorphic to the Baily-Borel line bundle $\mathcal{L}^{-1}_{\Gamma_{K3,2d}}$.*

For detail proof of Lemma 28 see Proposition **6.9.** of [19].

Theorem 29. *When the degree of the polarization is such that there is a single zero-dimensional cusp at the boundary of the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$ of $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$, then the function $\|\omega_{n,2d}\|^2_{L^2}$ on $\mathfrak{h}_{2,19}$ is a modular form of weight -2 with respect $\Gamma_{K3,2d}$ and it had no zeroes on $\mathfrak{h}_{2,19}$.*

PROOF: Theorem 29 follows directly from Lemma 28 and the definition of the normalized holomorphic form. ■.

Now we can conclude that when the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0 the function

$$|f_{K3,\omega_{n,2d}}|^2=\left(\frac{\det\Delta_{(X,e)}}{\|\omega_{n,2de}\|^2_{L^2}}\right)$$

is a modular function of weight 2 with respect to $\Gamma_{K3,2d}$ and its zero set is supported by $\mathcal{D}_{K3,2d}=(\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19})\setminus\mathcal{M}^{2d}_{K3,pa}$. Notice that $f_{K3,\omega_{n,2d}}$ is a holomorphic automorphic form defined up to a character χ of the group $\Gamma_{K3,2d}/[\Gamma_{K3,2d},\Gamma_{K3,2d}]$. A Theorem of Kazhdan states that this group is finite. See [11]. From here we conclude that $(f_{K3,\omega_{n,2d}})^N$, where $N=\#(\Gamma_{K3,2d}/[\Gamma_{K3,2d},\Gamma_{K3,2d}])$ will be an automorphic form on $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ whose zero set is supported by $\mathcal{D}_{K3,2d}$, where $\mathcal{D}_{K3,2d}$ is the complement of $\mathcal{M}^{2d}_{K3,pa}$ in $(\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19})$, i.e. $\mathcal{D}_{K3,2d}=(\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19})\ominus\mathcal{M}^{2d}_{K3,pa}$. Thus Theorem 21 is proved. ■.

We note that the case d=2 is the main point of consideration in [8]. In [8] Borcherds constructed an automorphic form for degree two polarization K3 surfaces, whose zero set is supported by the discriminant locus.

6. Uniform Bounds.

6.1. Uniform Bounds for K3 Surfaces. Given a family of polarized, algebraic K3 or Enriques surfaces fibred over a curve C, we can give a short argument determining the number of singular fibres.

Theorem 30. *Let C be an algebraic curve of genus g and $\pi:Y\to C$ be a three dimensional projective non-singular variety such that for every $t\in C$, $\pi^{-1}(t)=X_t$ is a non-singular K3 surface. Suppose that on Y we have a polarization class H such that $H|_{X_t}=e$ and <e,e>=2d, where d is any positive integer. Let m_∞ be the number of points on C for which the local monodromy operator is of infinite order. Then the number of singular fibres of π is less or equal to $2g-2+m_\infty$.*

PROOF: Consider the relative dualizing line bundle $\mathcal{K}_{\mathcal{X}/\mathcal{M}^{2d}_{K3,ppa}}$ of the universal family of marked polarized algebraic

$$\pi : \mathcal{X}^{2d} \rightarrow \mathcal{M}^{2d}_{K3,mpa} = \mathfrak{h}_{2,19}.$$

On $\pi_*\mathcal{K}_{\mathcal{X}/\mathcal{M}^{2d}_{K3,mpa}}$ we have a natural metric. Indeed the fibres of the line bundle $\pi_*\mathcal{K}_{\mathcal{X}/\mathcal{M}^{2d}_{K3,mpa}}$ over $\tau \in \mathcal{M}^{2d}_{K3,mpa}$ is $H^0(X_\tau, \Omega^2_{\Xi_\tau})$. So the natural metric will be

$$\|\cdot\|^2 = \langle \omega_\tau, \omega_\tau \rangle = \int_{X_\tau} \omega_\tau \wedge \overline{\omega_\tau}.$$

In [43] it is shown that one has the formula $c_1(\|\cdot\|^2) = -\mu_B$ where μ_B is the form associated with the complete Bergman metric on $\mathfrak{h}_{2,19}$. Hence, $-c_1(\|\cdot\|)$ defines a compete metric on $C \setminus E_\infty$ where E_∞ is the set of points of C around which the local monodromy is infinite. Note that $-c_1(\|\cdot\|)$ when restricted to $C \setminus E_\infty$ is integrable. This is so since the periods

$$\left(..., \int_{\gamma_i} \omega_t, ..\right)$$

are solutions of ordinary differential equations with regular solutions, hence have logarithmic growth near E_∞ and

$$\langle \omega_\tau, \omega_\tau \rangle = \left(..., \int_{\gamma_i} \omega_t, ..\right) (\langle \gamma_i, \gamma_j \rangle) \left(..., \overline{\int_{\gamma_i} \omega_t}, ..\right)^t.$$

By the Gauss-Bonnet theorem, we have

$$\int_{C \setminus E_\infty} -c_1(\|\cdot\|) = -\chi(C \setminus E_\infty) = 2g - 2 + m_\infty.$$

From Borcherds's result proved in [8], that one can always construct a holomorphic automorphic form for any polarization class e on $\Gamma_{2d} \backslash \mathfrak{h}_{2,19}$, whose zero set $\mathcal{H}_{K3,2d}$ contains the support of the discriminant locus $\mathcal{D}^{2d}_{K3}$, we observe that the number of points on C corresponding to singular fibres is less or equal to $-\chi(C \backslash E)$ since

$$\langle \mathcal{H}_{K3,2d}, C \rangle = \int_{C \setminus E_\infty} -c_1(\|\cdot\|) = -\chi(C \backslash E).$$

This proves Theorem 30. ■.

6.2. Uniform Bounds for Enriques Surfaces.

Theorem 31. *Let $\mathcal{Y} \rightarrow C$ be a family of* $\mathbf{H}$ *marked Enriques surface over the algebraic curve C. (The* $\mathbf{H}$ *marking is defined in Definition 14.) Let m_∞ be the number of points on C for which the local monodromy operator is of infinite order. Then the number of singular fibres of π is less or equal to $2(2g - 2 + m_\infty)$.*

PROOF: The proof of Theorem 31 is the same as the proof of Theorem 30 by taking into account Remark 19 which states that the discriminant locus in the moduli space $\widetilde{\Gamma}_{Enr} \backslash \Omega_{Enr} = \mathcal{M}_{Enr,\mathbf{H}}$ consists of $\mathcal{D}_+$ and $\mathcal{D}_-$ and $\mathcal{L}^{-1}_{\widetilde{\Gamma}_{Enr}} = \chi^+ \otimes \mathcal{O}(\mathcal{D}_+)$ and $\mathcal{L}^{-1}_{\widetilde{\Gamma}_{Enr}} = \chi^- \otimes \mathcal{O}(\mathcal{D}_-)$. Here χ^+ and χ^- are characters of the finite group $\widetilde{\Gamma}_{Enr}/[\widetilde{\Gamma}_{Enr}, \widetilde{\Gamma}_{Enr}]$. Thus Theorem 31 is proved. ■.

7. Finiteness Theorems.

7.1. K3 Surfaces. Given the structural results from **Section 2** and the construction of an automorphic form from **Section 4,** we now can prove finiteness for certain families of K3 surfaces. As before, the K3 surface is assumed to have a polarization of degree 2d such that the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0. We know from the result of Horikawa that for degree 2 K3 surfaces the Baily-Borel compactification $\overline{\Gamma_{K3,2}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0.

Definition 32. *Let C be a fixed, non-singular algebraic curve, and let E$\neq\emptyset$ and E be a fixed effective divisor on C such that all the points in E have multiplicity 1. Define Sh(C,E,{K3,2d}) to be the set of all isomorphism classes of three dimensional projective, polarized varieties which admit fibration over C such that any fibre over C \E is a non-singular K3 surface for which the induced polarization is such that the Baily Borel compactification of the coarse moduli space contains only one cusp of dimension zero.*

Theorem 33. *Let C be an algebraic curve and let $\pi : Y \to C$ be a three dimensional projective non-singular variety such that for every $t \in C$, $\pi^{-1}(t) = X_t$ is a non-singular K3 surface. Suppose that on Y we have a polarization class H such that $H\,|_{X_t} = e$ and <e,e>=2d is such that the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0. Then the family $\pi : Y \to C$ is isotrivial.*

PROOF: From the versal properties of the moduli space $\mathcal{M}^{2d}_{K3,ppa}$ (See [37]) it follows that we have a map

$$p : C \to \mathcal{M}^{2d}_{K3,ppa} = \Gamma_{2d}\backslash\mathfrak{h}_{2,19} \subset \overline{\mathcal{M}^{2d}_{K3,ppa}}$$

such that $\overline{p(C)} \cap \overline{\mathcal{D}^{2d}_{K3}} = \emptyset$. So we deduce that $p(C)$ is contained in the complement of $\overline{\mathcal{D}^{2d}_{K3}}$ in $\overline{\mathcal{M}^{2d}_{K3,ppa}}$, where $\overline{\mathcal{M}^{2d}_{K3,ppa}}$ is the Baily-Borel compactification of $\mathcal{M}^{2d}_{K3,ppa} = \Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ and $\overline{\mathcal{D}^{2d}_{K3}}$ is the closure of $\mathcal{D}^{2d}_{K3}$ in $\overline{\mathcal{M}^{2d}_{K3,ppa}}$. Since $\overline{\mathcal{D}^{2d}_{K3}}$ is an ample divisor, we deduced that the complement of $\overline{\mathcal{D}^{2d}_{K3}}$ in $\overline{\mathcal{M}^{2d}_{K3,ppa}}$ is an affine variety. We deduce that $p(C)$ must be a point, since $p(C)$ is a projective variety in the affine variety $\overline{\mathcal{M}^{2d}_{K3,ppa}} \ominus \overline{\mathcal{D}^{2d}_{K3}}$. This proved Theorem 33. ■.

Theorem 34. *Suppose that we fix an algebraic curve C, a divisor E$\neq\emptyset$ on C as in Definition 32 and degree of polarization <e,e>=2d such that the Baily-Borel compactification $\overline{\Gamma_{K3,2d}\backslash\mathfrak{H}_{2,19}}$ contains only one cusp of dimension 0, then the set Sh(C,E,{K3,2d}) is finite.*

PROOF: The proof will be done in two steps. **Step1.** We will prove that there exists a holomorphic map p:$C \to \overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}} = \overline{\mathcal{M}^{2d}_{K3,ppa}}$. **Step2.** We will apply Theorem 6 to the pair C and $\overline{\mathcal{M}^{2d}_{K3,ppa}}$ and the ample divisor $\overline{\mathcal{D}^{2d}_{K3}} \subset \overline{\mathcal{M}^{2d}_{K3,ppa}}$ taking into account that by Theorem 30 p(C) have a bounded volume to conclude Theorem 34.

Proposition 35. *There exists a holomorphic map p:$C \to \overline{\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}}$. such that $\overline{p}^*\overline{\mathcal{D}^{2d}_{K3}} =$ E.*

PROOF: From the versal properties of the moduli space $\Gamma_{K3,2d}\backslash\mathfrak{h}_{2,19}$ of pseudo-polarized algebraic K3 surfaces it follows that we have a map

$$p : C \setminus E \to \Gamma_{K3,2d} \backslash \mathfrak{h}_{2,19} \subset \overline{\Gamma_{K3,2d} \backslash \mathfrak{h}_{2,19}} = \overline{\mathcal{M}^{2d}_{K3,ppa}}$$

such that $p(C \setminus E) \cap \mathcal{D}^{2d}_{K3} = \emptyset$. (See [37].) Let us denote by E_f those points around which the monodromy have a finite order and by E_∞ those points around which the monodromy is of infinite order. From Theorem 9.5. of Ph. Griffith proved in [16] we can conclude that the map p can be prolonged through C$\backslash E_\infty$, i.e. we have $p : C \setminus E_\infty \to \Gamma_{K3,2d} \backslash \mathfrak{h}_{2,19}$ and p$(E_f) \subset \mathcal{D}^{2d}_{K3}$. Borel proved in [4]:

Theorem 36. *Let $\mathcal{S}$ be a bounded, symmetric domain and $\Gamma \subset Aut(\mathcal{S})$ an arithmetically defined discrete group of automorphisms. Let $\mathcal{U} := \Gamma \backslash \mathcal{S}$. Let p:$D^* \to \mathcal{U}$ be a holomorphic map from the punctured Disk D^* to the locally symmetric space $\mathcal{U}$, then p can be extended to a holomorphic map $\overline{p} : D \to \overline{\mathcal{U}}$, where $\overline{\mathcal{U}}$ is the Baily-Borel compactification of $\mathcal{U}$.*

Theorem 36 implies that there is a holomorphic map $\overline{p} : C \to \overline{\Gamma_{K3,2d} \backslash \mathfrak{h}_{2,19}}$ such that $Supp\left((\overline{p})^*(\overline{\mathcal{D}^{2d}_{K3}})\right) = E$. Proposition 35 is proved. ■.

For the curve C the map $\overline{p} : C \to \overline{\Gamma_{K3,2d} \backslash \mathfrak{h}_{2,19}}$ will be either finite map onto $\overline{p}(C)$ or it will be a map to a point. The last possibility is impossible since this will mean that the family $\pi : Y \to C$ is isotrivial. We assumed that this is not the case. From here and Theorem 30 we can see that conditions **a, b** and **c** of Theorem 6 are satisfied so we conclude that Sh(S,E,{K3,2d}) is finite. Theorem 34 is proved. ■.

7.2. Finiteness Theorems for S-K3 Surfaces. Following [8] we define an S-K3 surface X for some Lorenzian lattice S$\subset \Lambda_{K3}$ of rank ≥ 1 is a K3 surface with a fixed primitive embedding of S into the Picard group such that the image of S contains a pseudo-ample class. (A pseudo-ample class is a class D such that $\langle D, D\rangle > 0$ and $\langle D, C\rangle \geq 0$ for all curves C on the K3 surface X.

It follows from the surjectivity of the period map that the moduli space of S-K3 surfaces is isomorphic to $\Gamma_S \backslash \mathfrak{h}_{2,20-rkS}$, where Γ_S is an arithmetic group acting on $\mathfrak{h}_{2,20-rkS}$. See [44]. We can define in $\Gamma_S \backslash \mathfrak{h}_{2,20-rkS}$ the discriminant locus as in the case of polarized algebraic K3 surfaces. We define T_S as follows: $T_S := \{u \in \Lambda_{K3} | \langle u, S\rangle = 0\}$. Define the set $\Delta(S) : \Delta(S) := \{\delta \in T_S | \langle \delta, \delta\rangle = -2\}$. Remember that $\mathfrak{h}_{2,20-rkS}$ is one of the component of the set $\{u \in \mathbb{P}(T_S \otimes \mathbb{C}) | \langle u, u\rangle = 0 \& \langle u, u\rangle > 0\}$. For each $\delta \in \Delta_S$, define the hyperplane $H(\delta) = \{u \in \mathbb{P}(T_e \otimes \mathbb{C}) | \langle u, \delta\rangle = 0\}$. Let $\mathcal{H}_S = \bigcup_{\delta \in \Delta(S)} (H(\delta) \cap \mathfrak{h}_{2,20-rkS})$. Set $\mathcal{D}_S := \Gamma_S \backslash \mathcal{H}_S$.

Definition 37. *Let us define Sh(C,E,(K3,S)) as the set of families of algebraic S-K3 surfaces up to isomorphisms over the algebraic curve C such that for each t$\in$C$\backslash$E, $\pi^{-1}(t)$ is a non-singular S-K3 surface and for each t$\in$E, $\pi^{-1}(t)$ is a singular surface.*

Applying the same arguments as in the previous Section we obtain the following Theorem:

Theorem 38. *Suppose that we fix an algebraic curve C, a divisor E$\neq \emptyset$ on C as in Definition 37 and suppose that on $\mathfrak{h}_{2,20-rkS}$ there exists a holomorphic automorphic Φ_S form such that the supporter of the zero set of Φ_S is exactly $\mathcal{D}_S$, then the set Sh(C,E,{K3,S}) is finite.*

Borcherds and Nikulin found some lattices S as NS groups of K3 surfaces for which one can construct a holomorphic automorphic Φ_S form such that the supporter of the zero set of Φ_S is exactly $\mathcal{H}_S$. See [8].

7.3. Finiteness Theorems for Enriques Surfaces.

Definition 39. *Let C be a fixed, non-singular algebraic curve, and let E be a fixed effective divisor on C such that all the points in E have multiplicity 1. Define Sh(C,E,Enr) to be the set of all isomorphism classes of three dimensional projective, polarized varieties which admit fibration over C such that the fibres over $C \setminus E$ are non-singular Enriques surfaces for which the induced polarization is of degree 2d for d>1 or they are* **H** *polarized.*

Theorem 40. *Let C be an algebraic curve and let $\pi : Y \to C$ be a three dimensional projective non-singular variety such that for every $t \in C$, $\pi^{-1}(t) = X_t$ is a non-singular Enriques surface. Suppose that on Y we have a polarization class H such that $H|_{X_t} = e$ and <e,e>=2d for d>1 or all the fibres are* **H** *marked. Then the family $\pi : Y \to C$ is isotrivial.*

PROOF: We will define the groups $\Gamma_{Enr,2d}$ for $d > 1$ as follows:

Definition 41. *Let $e \in \Lambda_{K3}$ be a primitive element such that $< e, e >= 4d$ for $d > 1$, then*

$$\Gamma_{Enr,2d} := rest_{\Lambda^-_{K3}}\{g \in Aut(\Lambda_{K3}) | g \circ \rho = \rho \circ g \text{ and } g(e) = e\}.$$

Clearly $\Gamma_{Enr,2d}$ is a subgroup of finite index in Γ_{Enr} as defined in Definition 13.

First we will define the moduli space $\mathcal{M}_{Enr,2d}$ of polarized Enriques surfaces with degree of polarization 2d>2. We already defined

$$\mathcal{M}_{Enr,\mathbf{H}} = \left(\widetilde{\Gamma}_{Enr} \backslash \mathfrak{h}_{2,10}\right) \ominus (\mathcal{D}_+) \cup \mathcal{D}_-).$$

It was proved in [10] that $\mathcal{M}_{Enr,\mathbf{H}}$ is the coarse moduli space of **H** marked Enriques surfaces.

Definition 42. *We will define $\mathcal{M}_{Enr,2d}$ as follows:*

$$\mathcal{M}_{Enr,2d} := (\Gamma_{Enr,2d} \backslash \mathfrak{h}_{2,10}) \ominus (\pi^{-1}_{2d}(\mathcal{D}_{E,+}) \cup \pi^{-1}_{2d}(\mathcal{D}_{E,-})),$$

where $\pi_{2d} : \Gamma_{Enr,2d} \backslash \mathfrak{h}_{2,10} \to \Gamma_{Enr} \backslash \mathfrak{h}_{2,10}$ is the natural map and $\pi_{\mathbf{H}} : \widetilde{\Gamma}_{Enr} \backslash \mathfrak{h}_{2,10} \to \Gamma_{Enr} \backslash \mathfrak{h}_{2,10}$. $\mathcal{D}_{\pm}$ are defined in Definition 16 and $\mathcal{D}_{E,\pm} = (\pi_{\mathbf{H}})_* (\mathcal{D}_{\pm})$.

Global Torelli Theorem for Enriques surfaces implies that $\mathcal{M}_{Enr,2d}$ is coarse moduli space of Enriques surfaces with 2d polarization. It follows from the versal properties of the coarse moduli space $\mathcal{M}_{Enr,2d}$ of polarized Enriques surfaces for $d > 1$ that we have a holomorphic map $p : C \to \mathcal{M}_{Enr,2d}$. Since $\pi_{\mathbf{H}} : \widetilde{\Gamma}_{Enr} \backslash \mathfrak{h}_{2,10} \to \Gamma_{Enr} \backslash \mathfrak{h}_{2,10}$ is a finite map, then the image of an affine open set or quasi affine are also affine or quasi-affine. We know that according to [9]

$$\left(\widetilde{\Gamma}_{Enr} \backslash \mathfrak{h}_{2,10}\right) \ominus (\mathcal{D}_+) \cup \mathcal{D}_-)$$

is quasi-affine. From here we obtain that $\mathcal{M}_{Enr,2d}$ is quasi-affine. So the map $p : C \to \mathcal{M}_{Enr,2d}$ is a map to a point. The same arguments are applied when we consider **H** marked Enriques surfaces. Thus we proved Theorem 40. ■.

Theorem 43. *Suppose that we fix and algebraic curve C, a divisor E$\neq\emptyset$ as in Definition 39, and degree of polarization <e,e>=2d. Then the set Sh(C,E,Enr) is finite.*

PROOF: The proof of Theorem 43 is reduced to the case of $\mathbf{H}$ marked Enriques surfaces and then the proof goes exactly in the same way as in the case of pseudo polarized algebraic K3 surfaces whose moduli space has a unique cusp of dimension zero in the Baily Borel compactification.

From the versal properties of the coarse moduli space of polarized Enriques surfaces $\mathcal{M}_{Enr,2d}$ and by using the results of Griffiths and Borel and the properties of the Baily-Borel compactification, we get a holomorphic map:

$$p : C \to \overline{\Gamma_{Enr,2d}\backslash\mathfrak{h}_{2,19}}$$

where $\overline{\Gamma_{Enr,2d}\backslash\mathfrak{h}_{2,19}}$ is the Baily-Borel compactification of $\Gamma_{Enr,2d}\backslash\mathfrak{h}_{2,19}$. Since $\Gamma_{Enr,2d}$ is a subgroup of finite index in Γ_{Enr} we obtain a holomorphic map $p_1 : C \to \overline{\Gamma_{Enr}\backslash\mathfrak{h}_{2,19}}$. We have a finite fixed map according to [5] and [6]: $\pi_d : \overline{\Gamma_{Enr,2d}\backslash\mathfrak{h}_{2,19}} \to \overline{\Gamma_{Enr}\backslash\mathfrak{h}_{2,19}}$. We also know that $\widetilde{\Gamma}_{Enr}$ is a subgroup of finite index in Γ_{Enr}. So we have a finite map: $\pi_{\mathbf{H}} : \overline{\widetilde{\Gamma}_{Enr}\backslash\mathfrak{h}_{2,19}} \to \overline{\Gamma_{Enr}\backslash\mathfrak{h}_{2,19}}$.

After taking a finite cover

$$\psi : \widetilde{C} = C \times_{p_1(C)} \pi_{\mathbf{H}}^{-1}(p_1(C)) \to C$$

of a degree less or equal to $\deg \pi_{\mathbf{H}}$, we will get a finite holomorphic map

$$\widetilde{p} : \widetilde{C} \to \widetilde{p}(\widetilde{C}) \subset \overline{\widetilde{\Gamma}_{Enr}\backslash\mathfrak{h}_{2,19}}.$$

Over $\widetilde{C}$ we have a family of Enriques surfaces with $\mathbf{H}$ polaization and discriminant locus $\widetilde{E} = \psi^{-1}(E)$. We can apply now Theorem 31 to get a bound on the number of points in $\widetilde{E}$ and the volume of the image of $\widetilde{p}(C)$ in $\overline{\widetilde{\Gamma}_{Enr}\backslash\mathfrak{h}_{2,19}}$. Since the degree of the map ψ is fixed we get a bound on the volume of the image $p_1(C)$ in $\overline{\Gamma_{Enr}\backslash\mathfrak{h}_{2,19}}$ and respectively of the volume of the image $p(C)$ in $\overline{\Gamma_{Enr,2d}\backslash\mathfrak{h}_{2,19}}$. Notice also that from Remark 19 we conclude that $(\pi_d)^*(\pi_{\mathbf{H}})_*(\mathcal{D}_+ + \mathcal{D}_-)$ is an ample divisor. Since $p(E) \subset (\pi_d)^*(\pi_{\mathbf{H}})_*(\mathcal{D}_+ + \mathcal{D}_-)$, Theorem 6 implies that the set $\mathrm{Sh}(C, E, Enr)$ is finite by repeating the arguments of Theorem 34. From here Theorem 43 follows directly. ■.

8. Isotriviality.

Definition 44. *We will say that a family of algebraic varieties $X \to Y$ is an isotrivial family if there exists a finite map $\phi : Y_1 \to Y$ such that the family $X \times_Y Y_1 \to Y_1$ is a trivial one, i.e. the family $X \times_Y Y_1 \to Y_1$ is isomorphic to $Y_1 \times Z$.*

Theorem 45. *Suppose that $\mathcal{M}$ is the coarse moduli space of polarized algebraic varieties Z. Suppose that $\mathcal{M}$ is a quasi projective variety. Let $\overline{\mathcal{M}}$ be some projective compactification of $\mathcal{M}$ such that $\overline{\mathcal{M}} \ominus \mathcal{M} = \mathcal{D}$ is a divisor with normal crossings. If $\mathcal{D}$ supports an ample divisor, then any family $\mathcal{Z} \to C$ of algebraic polarized varieties Z over a projective variety C without singular fibres is isotrivial.*

PROOF: The proof is obvious since the condition that $\mathcal{D}$ supports an ample divisor implies that $\mathcal{M}$ is an affine or quasi-affine. On the other hand side from the versal properties of the course moduli space we deduce that there is a map $p : C \to \mathcal{M}$. Since C is a projective variety, then $p(C)$ must be a point. ■. See also [8].

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